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Good and Bad Shapes for Symmetric Group Representations

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INTRODUCTION

It is well known that if K is a field of characteristic zero, all the irreducible representations of S_n may be obtained from proper Young diagrams, that is, arrays in the shape of a right-angled triangle. In [1] a method is given for finding the composition factors of modules corresponding to skew shapes. Bases are also known for skew modules. Less is known about the module corresponding to a general array of nodes in rows and columns, and it is of interest to determine when such an array is equivalent to a proper or skew shape.

We begin with an arbitrary array of n nodes in rows and columns. The numbers $1, \ldots, n$ may be assigned to the nodes in any order to give a tableau x. If K is any field, a representation module for the symmetric group S_n is provided by the ideal $KS_n \varepsilon C(x) \iota R(x)$ [2], where $\varepsilon C(x)$ is the alternating sum of all permutations which rearrange nodes inside columns, and $\iota R(x)$ is the sum of permutations which rearrange nodes within rows. Clearly the ideal is unchanged if whole rows or whole columns are interchanged.

In [3] *bad shapes* were defined as arrays which could not be made into a proper or skew diagram by any interchange of rows or columns. In this paper we give an algorithm to form a proper or skew diagram if one exists. The algorithm terminates when either all the rows have been placed in a suitable order or it has been proved that the task is impossible. For small n, it may be worthwhile to look briefly at the array and see if any of the causes of failure listed in Section 3 are apparent, before commencing the construction.

If the array can be rearranged to form a proper Young diagram the shape is uniquely determined. For skew diagrams, a rotation through 180° is always possible. Also if any part of the diagram has no columns in com-

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mon with the rest, it can be placed at either end. It will be shown that these are the only possible variations. The paper ends with a worked example.

We require a fixed labeling for the columns in the initial array, and for each row its length, that is, the number of nodes it contains, and a list of the columns in which its nodes lie. Denote by $\mathscr{C}(R)$ the set of columns having nodes in row R. Set $L(R)$ be the length of R and $\mathrm{Ov}(R, S)$, the overlap of rows R and S, that is, the number of columns they have in common. It is clear that if there is a set of two or more rows having nodes in exactly the same columns, they must be placed together in any skew diagram, and their order within the set makes no difference to the final shape. Such a set of identical rows will be treated as a single row in the description that follows.

2. Structural Properties

The following properties of skew diagrams are easily verified:

2.1. *If any number of rows are removed from a skew shape, the remaining rows can be compacted without reordering to give a new skew shape.* The existence of a small set of rows which cannot be put together in a skew shape is therefore sufficient to show that no suitable arrangement of the whole array exists.

2.2. *If A is any row of a skew diagram and $\mathscr{C}(B) \subseteq \mathscr{C}(A)$, then B will be closer to A than any row on the same side of A having nodes not in $\mathscr{C}(A)$.*

2.3. *Let A, B, C be rows in a skew diagram. If neither of the intersections $\mathscr{C}(A) \cap \mathscr{C}(B)$, $\mathscr{C}(A) \cap \mathscr{C}(C)$ is contained in the other, B and C lie on opposite sides of A, and the columns are arranged so that $\mathscr{C}(A) \cap \mathscr{C}(B)$ and $\mathscr{C}(A) \cap \mathscr{C}(C)$ are at opposite ends of A.*

2.4. *If $\mathrm{Ov}(A, B) = \mathrm{Ov}(A, C)$ then either $\mathscr{C}(A) \cap \mathscr{C}(B) = \mathscr{C}(A) \cap \mathscr{C}(C)$ or B and C lie on opposite sides of A.* This follows from 2.3.

2.5. *In a skew diagram the rows having greatest overlap with A will be closest to A on either side.* This determines the order in which rows may be added to a diagram.

2.6. *If A, B, C are three rows of a skew diagram placed in order, then* (i) $\mathscr{C}(B) \supseteq \mathscr{C}(A) \cap \mathscr{C}(C)$; (ii) *If $\mathscr{C}(A) \cap \mathscr{C}(C)$ is not empty, $\mathscr{C}(B) \subseteq \mathscr{C}(A) \cup \mathscr{C}(C)$.*

2.7. *If three rows can be arranged in order to form a skew diagram, another skew diagram can be formed by placing them in reverse order, and this is equivalent to a rotation through* 180°. *Any other permutation of the rows gives a non-skew shape.*

3. Causes of Failure

3.1. Lemma. *Rows A, B, C cannot be placed in a skew diagram if $\mathscr{C}(A)\cap\mathscr{C}(B)$, $\mathscr{C}(B)\cap\mathscr{C}(C)$, and $\mathscr{C}(A)\cap\mathscr{C}(C)$ are all non-empty, and none of these intersections is contained in any other.*

Proof. Suppose such a diagram can be formed. If neither of $\mathscr{C}(A)\cap\mathscr{C}(B)$, $\mathscr{C}(B)\cap\mathscr{C}(C)$ is contained in the other then by 2.3 (ii) A and C are on opposite sides of B and neither of them spans the whole length of B. Then $\mathscr{C}(A)\cap\mathscr{C}(C)$ is contained in $\mathscr{C}(B)$ and hence in $\mathscr{C}(A)\cap\mathscr{C}(B)$—contradiction.

3.2. Lemma. *Rows A, B, C cannot form a skew diagram if $\mathscr{C}(A)\cap\mathscr{C}(B)\cap\mathscr{C}(C)$ is not empty and each of $\mathscr{C}(A)$, $\mathscr{C}(B)$. $\mathscr{C}(C)$ contains at least one column which is not in either of the others.*

Proof. By permuting, columns A and B can be placed with $\mathscr{C}(A)\cap\mathscr{C}(B)$ in the middle. The nodes unique to A and B will then appear at either end. To preserve columns, C would then have to be placed with some nodes in $\mathscr{C}(A)\cap\mathscr{C}(B)$ and others detached from them at one end, which cannot give a skew shape.

3.3. Lemma. *Rows A, B, C cannot be placed in a skew diagram with B between A and C if $\mathscr{C}(A)\cap\mathscr{C}(C)$ is not empty and $\mathscr{C}(B)\nsubseteq\mathscr{C}(A)\cup\mathscr{C}(C)$.*

This follows immediately from 2.6 (ii).

3.4. Lemma. *B and C cannot be placed in a skew diagram on the same side of A if $\mathscr{C}(B)\subset\mathscr{C}(A)$ and $\mathscr{C}(C)\nsubseteq\mathscr{C}(A)$ and C overlaps with A by more nodes than B does.*

This follows from 2.2 and 2.5.

3.5. Lemma. *Rows A, B, C, D cannot form a skew diagram if none of the intersections $\mathscr{C}(A)\cap\mathscr{C}(B)$, $\mathscr{C}(A)\cap\mathscr{C}(C)$, $\mathscr{C}(A)\cap\mathscr{C}(D)$ is contained in any other.*

Proof. By 2.3 (ii) B and C must lie on opposite sides of A in any skew diagram. Similarly, D would have to lie on the opposite side of A from both B and C, which is impossible.

3.6. Corollary. *A, B, C, D cannot be placed in a skew diagram if $\mathscr{C}(A)\cap\mathscr{C}(B)$, $\mathscr{C}(A)\cap\mathscr{C}(C)$, and $\mathscr{C}(A)\cap\mathscr{C}(D)$ all contain the same number of columns and no two of the intersections are identical.*

4. Building a Skew Diagram

The diagram is built outwards in both directions from the longest row, rearranging columns as necessary. A row is said to be *deposited* when it has been placed in the diagram. If two or more rows have nodes in exactly the same columns, they are deposited together and treated as one row in the subsequent description. If at any stage a diagram is obtained which has no columns in common with the remaining rows, this can be placed at the bottom left of the final shape. The algorithm then recommences on the remaining rows.

4.1. *Construction.* Select the longest row R. If there is more than one row of this length, any one may be chosen. Rearrange columns to bring the nodes of R together and deposit any rows identical to R. Let X be the set of rows whose nodes all lie in $\mathscr{C}(R)$ and Y the set having some nodes in $\mathscr{C}(R)$ and some outside. In accordance with 2.2 and 2.5 we deposit the rows of X first, then those of Y, and finally any rows having no overlap with R. If X is empty, proceed to 4.6. Otherwise, choose R_1 from X to be as long as possible. If X contains more than two non-identical rows of this length, no skew diagram can be formed, by 3.6. By 2.2 and 2.5, R_1 must be deposited next to R in any skew diagram. Form a sequence of rows of X, $R_1, ..., R_s$, such that for each i, $2 \leqslant i \leqslant s$, $\mathscr{C}(R_i) \subseteq \mathscr{C}(R_{i-1})$ and R_i is as long as possible, and continuing until no further rows can be added. By rearranging columns these rows can be formed into a (left-justified) proper diagram below R.

4.2. Lemma. *No skew diagram can be formed if for any $i > 1$ there are two distinct choices for R_i.*

If A, B are both possible choices for R_i, $\mathrm{Ov}(A, R) = \mathrm{Ov}(B, R)$ and since they are not identical neither of their intersections with R is contained in the other. So by 2.3 they can only be placed on opposite sides of R, and at opposite ends. But $\mathscr{C}(A)$ and $\mathscr{C}(B) \subseteq \mathscr{C}(R_1)$ and $L(R_1) < L(R)$ so at least one node of R lies to the right of all columns of R_1 and is not in $\mathscr{C}(A)$ or $\mathscr{C}(B)$. Hence A and B cannot be placed at opposite ends of R, so they cannot both be deposited.

4.3. Lemma. *When* 4.1 *is completed, no more rows of X can be deposited below R.*

Suppose $\mathscr{C}(S) \subset \mathscr{C}(R)$ and S has not been deposited. Then $\mathscr{C}(S) \subseteq \mathscr{C}(R_s)$. Let R_i be the highest row such that $\mathscr{C}(S) \subseteq \mathscr{C}(R_i)$. Then $L(S) < L(R_i)$ otherwise S would have been chosen instead of R_i, so $\mathscr{C}(R_i) \subseteq \mathscr{C}(S)$. By 2.3, S and R_i cannot be deposited on the same side of R.

The remaining rows of X must therefore be deposited, if at all, in a right-justified upper triangular shape above R, by the following method.

4.4. *Construction.* If X is now empty, proceed to 4.6. Otherwise, let T_1 be the longest remaining row in X. By 2.2 this must be deposited next to R if at all. By 2.4 and 4.1 if there are two choices for T_1 no diagram can be formed. Form a maximal sequence $T_1, ..., T_t$ such that $\mathscr{C}(T_i) \subseteq \mathscr{C}(T_{i-1})$. Clearly this must satisfy all the restrictions placed on $R_1, ..., R_s$ in order to be deposited above R in a triangular shape.

Also, because $R_1, ..., R_s$ are already deposited, an arbitrary rearrangement of columns is no longer possible.

4.5. LEMMA. *T_j cannot be deposited above R if for some i, $1 \leqslant i \leqslant s$, $\mathscr{C}(T_j) \cap \mathscr{C}(R_i)$ is non-empty and R contains one or more nodes not in $\mathscr{C}(T_j)$ or $\mathscr{C}(R_i)$.*

This follows immediately from 3.3. If any rows of X are left after $T_1, ..., T_t$ have been deposited, 4.3 shows that they cannot be placed above or below R, so no skew diagram is possible.

4.6. *Construction.* If Y is empty, the diagram so far is independent of the remaining rows, and the construction recommences from 4.1. Assume Y is not empty. If X was empty originally, R_1 has not yet been defined. In this case choose R_1, from Y to have maximum overlap with R, and deposit it below R. If T_1 has not yet been defined and there are any rows left in Y, choose T_1 so that $\mathscr{C}(T_1) \cap \mathscr{C}(R) \not\subseteq \mathscr{C}(R_1)$ and $\mathrm{Ov}(T_1, R)$ is maximal.

By 2.3, T_1 could not be placed on the same side of R as R_1. If T_1, R, R_1 satisfy 2.6, T_1 can be placed above R, otherwise no skew diagram is possible.

When R_1 and T_1 have been deposited, the directions for building outward are established. In the subsequent construction there is no essential difference between adding rows above and below the existing diagram and every condition imposed on R_i is matched by a condition on T_j, $i, j \geqslant 1$.

4.7. *Construction.* When 4.6 is completed, we have a skew diagram containing all the rows of X and at least one row above and below R, unless X was empty and Y contained at most one row, in which case R is at the top of the diagram. If the remaining rows can be added to the diagram they must clearly satisfy the following conditions:

(i) When R_i has been deposited, by 2.5, R_{i+1} must be chosen to have maximum overlap with R_i.

(ii) By 3.3, either $\mathscr{C}(R_i) \cap \mathscr{C}(R_{i+2})$ is empty or $\mathscr{C}(R_{i+1}) \subseteq \mathscr{C}(R_i) \cup \mathscr{C}(R_{i+2})$.

(iii) R_{i+1} has no nodes in columns which have already been deposited to the right of R_i.

(iv) All nodes of R_{i+1} which are not in $\mathscr{C}(R_i)$ are placed at the left-hand end of R_{i+1}.

It remains only to show that R_{i+1} is uniquely determined by R_i, and that no different choice of starting row could have produced a diagram containing all the rows.

4.8. THEOREM. *When R_i has been deposited, either R_{i+1} is uniquely determined, or no skew diagram exists.*

Proof. Suppose that for all $j \leqslant i$, R_j is uniquely determined, and there are two choices A, B for R_{i+1}, both satisfying 4.7 (i)–(iv). There are three possibilities:

(i) $\mathscr{C}(A) \subseteq \mathscr{C}(B)$. Then A can be deposited first, followed by B.

(ii) $\mathscr{C}(A) \cap \mathscr{C}(R_i) = \mathscr{C}(B) \cap \mathscr{C}(R_i)$ but neither of $\mathscr{C}(A)$, $\mathscr{C}(B)$ is contained in the other.

By 2.3, A and B cannot both be deposited below R_i. By hypothesis, neither of them can be deposited between R and R_i. $\mathscr{C}(B) \cap \mathscr{C}(R_i)$ is non-empty, and R contains nodes not in $\mathscr{C}(B) \cup \mathscr{C}(R_i)$, so by 2.6, B cannot be deposited above R, and by the same reasoning, neither can A. Hence no skew diagram can be formed.

(iii) $\mathscr{C}(A) \cap \mathscr{C}(R_i) \neq \mathscr{C}(B) \cap \mathscr{C}(R_i)$. By 2.3, A and B cannot be deposited on the same side of R_i. The proof then follows as in case (ii).

For the remainder of this section it is convenient to have a total ordering of rows. We set $R = R_0$ and $T_k = R_{(-k)}$.

4.9. THEOREM. *If as many rows as possible have been deposited according to* 4.1–4.7, *either all rows must be used up or no skew diagram can be formed.*

Proof. Suppose row L has not been deposited. Since independent rows can be deposited at either end, L must overlap with some row in the diagram. Let R_i be the highest such row. If R_i is the top row, L cannot be placed above it; otherwise L would have been deposited already.

If R_i is not the top row then $\mathscr{C}(L) \cap \mathscr{C}(R_{i-1})$ is empty, so neither of $\mathscr{C}(L) \cap \mathscr{C}(R_i)$, $\mathscr{C}(R_{i-1}) \cap \mathscr{C}(R_i)$ is contained in the other, and by 2.3, L must be deposited below R_i if at all. If $i \geqslant 0$, this is impossible by 4.8. It remains only to show that if i is negative L could not be inserted between R_i and R_0 even if a different starting row were chosen for building the diagram. By 4.8, when R_0 is taken as starting row, every row between R_0

TABLE I

	1	2	3	4	5	6	7	8
1		×		×			×	×
2	×							
3	×	×		×			×	
4			×			×		×
5	×	×		×			×	×
6		×				×		×
7	×				×			
8	×			×				
9	×			×				

and R_i is uniquely determined, and each row above R_0 is chosen to have maximum overlap with the row below it.

Suppose that by choosing a higher row to start with and working downwards, L could be deposited between R_j and R_{j+1} for some j, $i \leqslant j \leqslant -1$. Then by 2.6, $\mathscr{C}(L) \supseteq \mathscr{C}(R_j) \cap \mathscr{C}(R_{j+1})$, so $\mathscr{C}(L) \cap \mathscr{C}(R_{j+1}) \supseteq \mathscr{C}(R_j) \cap \mathscr{C}(R_{j+1})$; that is, L overlaps with R_{j+1} by at least as many nodes as R_j does. Also by 2.6, $\mathscr{C}(R_j) \cup (R_{j+1}) \supseteq \mathscr{C}(L)$, and these two conditions are sufficient for L to be placed between R_{j+1} and R_j when the diagram is built up from R_0. This contradicts the hypothesis that L was not deposited originally. Hence L cannot be deposited, and no skew diagram can be formed.

4.10. COROLLARY. *If an array can be rearranged to form a proper or skew diagram, the shape is unique, apart from the relative positions of non-overlapping parts, and possible rotations through* 180°.

4.11. EXAMPLE. Suppose the initial array is as in Table I. The numbering of rows and columns will be used to identify them in their new positions. Rows 8 and 9 are identical so they will be kept together throughout. Row 5 is the longest, having five nodes, so this is chosen as R and columns

TABLE II

	1	4	7	2	8
5	×	×	×	×	×
3	×	×	×	×	
8	×	×			
9	×	×			
2	×				

TABLE III

	5	1	4	7	2	8	6	3
$T_3 = 4$						×	×	×
$T_2 = 6$					×	×	×	
$T_1 = 1$			×	×	×	×		
$R = 5$		×	×	×	×	×		
$R_1 = 3$		×	×	×	×			
$R_2 = 8$		×	×					
$R_3 = 9$		×	×					
$R_4 = 2$		×						
$R_5 = 7$	×	×						

1, 2, 4, 7, 8 must be grouped together in some order. The other rows of X are 1, 2, 3, 8, 9. Of these, rows 1 and 3 both have four nodes, so either may be chosen as R_1, and this determines the orientation of the final shape.

Taking R_1 as row 3 leads to the sequence 5, 3, 8, 9, 2 with $\mathscr{C}(5) \supset \mathscr{C}(3) \supset \mathscr{C}(8) = \mathscr{C}(9) \supset \mathscr{C}(2)$ and these rows can be placed in order with suitable rearrangement of columns to give a proper Young diagram as shown in Table II.

Row 1 may now be placed above row 5 without rearrangement of columns, to form part of a skew diagram, so this is chosen as T_1. Construction 4.7 determines uniquely the order in which the remaining rows may be deposited, and in this example the algorithm succeeds, giving the skew diagram shown in Table III.

Now suppose an additional node is inserted in the original array in row 4, column 7. The algorithm proceeds as before until T_1 has been deposited, but there are two choices for T_2 since $\mathscr{C}(T_1) \cap \mathscr{C}(6) = \{2, 8\}$ and $\mathscr{C}(T_1) \cap \mathscr{C}(4) = \{7, 8\}$; moreover, columns 7, 2 could have been placed in either order up to this point. Hence by 4.8 no skew diagram exists. Failure could also have been predicted without tracing through the algorithm, by applying 3.1 to rows 1, 4, and 6.

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